SOLUTION OF A PLANE STEADY-STATE PROBLEM IN THE THEORY OF HEAT CONDUCTION FOR A BOUNDARY CONDITION OF THE THIRD KIND FOR REGIONS BOUNDED BY CYCLOIDAL CURVES

B. A. Vasil'ev

UDC 536.2

We demonstrate that the steady-state problem of the theory of heat conduction for regions bounded by cycloidal curves – releasing heat from the surface according to Newton's law – can be reduced through application of conformal mapping to the solution of equations in finite differences, solvable in terms of Bessel functions with one variable.

Plane steady-state problems in the theory of heat conduction are reduced to the solution of the Laplace equation [1]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ in } D \tag{1}$$

for the boundary condition

$$\frac{\partial u}{\partial n}+hu\Big|_{\Gamma}=f(p), \tag{2}$$

where h is a positive constant and f(p) is a specified function.

In the general case the boundary conditions (2) do not allow for effective application of the method of conformal mapping with respect to solution of the problem. However, as demonstrated in [3], we can isolate a special class of domains for which the method of conformal mapping onto a circle makes it possible to reduce the problem to the solution of finite-difference equations solvable in Bessel functions with many variables [4]. The domains treated in [3] represent a special case of the broader class of domain produced in the mapping of a unit circle with the aid of the function

$$W(\xi)=A\int_0^{\xi}[P_m(\xi)]^2\,d\xi, \tag{3}$$

where $\xi = \exp i\theta$; $W = x + iy$; A is an arbitrary constant; $P_m(\xi)$ is a polynomial of degree m with real coefficients, and $P_m(0) \neq 0$.

Of particular interest are the domains bounded by cycloidal curves [2], and these are produced through conformal mappings of the form

$$W(\xi)=A\int_0^{\xi}[\xi^m+\lambda]^2\,d\xi,\ m=1,\ 2,\ 3,\ \ldots, \tag{4}$$

where $\xi = \rho\exp i\theta$; $|\rho| \leq 1; -\pi \leq \theta \leq +\pi$; λ is a positive real diameter; A is an arbitrary constant. The geometric curves (4), corresponding to the unit circle in the ξ plane, have m lobes and are simple closed curves when $\lambda \geq \lambda_0^m$ and $\lambda_0 = 2/\sqrt{3}$.* The arc length of these curves is a rational function of the parameter

*When $\lambda = \lambda_0$ (m = 1), the contour has one singularity which represents a point of osculation for two parts of the contour.

Kalinin Polytechnic Institute, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 4, pp. 680-686, April, 1969. Original article submitted May 24, 1968.

$$S=\int_{-\pi}^{+\pi}\left|\frac{dW}{d\xi}\right|\Bigg|_{\rho=1} d\theta = A\int_{-\pi}^{+\pi}(1+2\lambda\cos m\theta+\lambda^2)\,d\theta = 2\pi A(1+\lambda^2). \tag{5}$$

From (4) and (5) it is easy to derive the equation for the curves in parametric form:

$$x=\frac{S\lambda^2}{2\pi(1+\lambda^2)}\left\{\cos\theta+\frac{2\cos(m+1)\theta}{\lambda(m+1)}+\frac{\cos(2m+1)\theta}{\lambda^2(2m+1)}\right\},$$
$$y=\frac{S\lambda^2}{2\pi(1+\lambda^2)}\left\{\sin\theta+\frac{2\sin(m+1)\theta}{\lambda(m+1)}+\frac{\sin(2m+1)\theta}{\lambda^2(2m+1)}\right\}, \tag{5a}$$

where $-\pi \le \theta \le +\pi$; $m = 1, 2, 3, \ldots, N$; S is the arc length; $\lambda \ge \lambda_0^m$ is a parameter.

1. Formulation of the Problem for a Circle and Reduction of this Problem to the Solution of Finite-Difference Equations

With the conformal mapping (4)

$$W(\xi)=\frac{S\lambda^2}{2\pi(1+\lambda^2)}\left\{\xi+\frac{2\xi^{m+1}}{\lambda(m+1)}+\frac{\xi^{2m+1}}{\lambda^2(2m+1)}\right\},\ m=1,2,3,\ldots,N, \tag{6}$$

Eq. (1) and boundary conditions (2) are brought to the form

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial^2 u}{\partial\theta^2}=0 \tag{7}$$

in a circle with the boundary condition

$$\frac{\partial u}{\partial\rho}+\frac{hS}{2\pi}\left(1+\frac{2\lambda}{1+\lambda^2}\cos m\theta\right)u\Bigg|_{\rho=1}=f_1(\theta), \tag{8}$$

where $f_1(\theta)$ is a specified function satisfying the Dirichlet conditions in the interval $-\pi \le \theta \le +\pi$.

Then

$$f_1(\theta)=\frac{b_0}{2}+\sum_{n=1}^{\infty}[b_n\cos n\theta+c_n\sin n\theta]. \tag{9}$$

The unique feature of this case, enabling us to derive an exact solution for the problem, rests in the fact that the coefficient

$$h(\theta)=h\left|\frac{dW}{d\xi}\right|\Bigg|_{\rho=1}=\frac{hS}{2\pi}\left(1+\frac{2\lambda}{1+\lambda^2}\cos m\theta\right) \tag{10}$$

is a trigonometric binomial

$$h(\theta)=a_0+a_1\cos m\theta,\ -\pi\leqslant\theta\leqslant+\pi, \tag{11}$$

where

$$a_0=\frac{hS}{2\pi};\ a_1=\frac{hS}{\pi}\,\frac{\lambda}{1+\lambda^2}.$$

We seek the solution for (7) and (8) in the form

$$u=\frac{A_0}{2}+\sum_{n=1}^{\infty}\rho^n[A_n\cos n\theta+B_n\sin n\theta]. \tag{12}$$

Substituting (12) into boundary condition (8), for the determination of the expansion coefficient we derive a system of difference equations

$$(n+a_0)A_n+\frac{a_1}{2}[A_{n+m}+A_{n-m}]=b_n,\ n=0,1,2,3,\ldots; \tag{13}$$

$$(n+a_0)B_n+\frac{a_1}{2}[B_{n+m}+B_{n-m}]=c_n,\ n=1,2,3,\ldots, \tag{14}$$

under the condition

$$A_n \underset{n\to\infty}{\to} 0, \qquad B_n \underset{n\to\infty}{\to} 0; \tag{15}$$

$$A_n = A_{-n}, \quad n = 1, 2, 3, \ldots, m; \tag{16}$$

$$B_n = -B_{-n}, \quad n = 0, 1, 2, \ldots, m-1. \tag{17}$$

Using the results of [3], it is not difficult to obtain a solution for (13) and (14) in the form

$$A_n = \bar{A}_n + \sum_{k=1}^{s}\left[\omega_k \cos\frac{2k-1}{m}n\pi + \tilde{\omega}_k \sin\frac{2k-1}{m}n\pi\right] J_{\frac{n+a_0}{m}}\left(\frac{a_1}{m}\right), \tag{18}$$

$$B_n = \bar{B}_n + \sum_{k=1}^{s}\left[\bar{\omega}_k \cos\frac{2k-1}{m}n\pi + \bar{\bar{\omega}}_k \sin\frac{2k-1}{m}n\pi\right] J_{\frac{n+a_0}{m}}\left(\frac{a_1}{m}\right), \tag{19}$$

where m = 2s or m = 2s−1, s = 1, 2, 3, . . . , N; ω_k; $\tilde{\omega}_k$; $\bar{\omega}_k$; and $\bar{\bar{\omega}}_k$ are arbitrary constants; $J_\nu(x)$ is a Bessel function of the first kind; $\bar{A}_n$ and $\bar{B}_n$ are particular solutions of (13) and (14), satisfying conditions (15). The particular solutions $\bar{A}_n$ and $\bar{B}_n$ can be presented in terms of the Green's function [3] in finite-difference equations

$$\bar{A}_n = \sum_{l=0}^{\infty} b_l g_{ln}; \quad \bar{B}_n = \sum_{l=1}^{\infty} c_l g_{ln}. \tag{20}$$

When the number for b_n and c_n is finite, the solutions $\bar{A}_n$ and $\bar{B}_n$ can be found from the recurrence relationships

$$\bar{A}_{n-m} = \frac{2}{a_1}\left[b_n - (n+a_0)\bar{A}_n\right] - \bar{A}_{n+m}, \tag{21}$$

where n = p; p−1; p−2; . . . ; 0, under the condition

$$b_{p+1} = b_{p+2} = \cdots = 0, \ \bar{A}_{p-m} = \frac{2b_p}{a_1}, \qquad \bar{A}_{p-m+1} = \bar{A}_{p-m+2} = \cdots = 0. \tag{22}$$

We have similar formulas for $\bar{B}_n$.

Having substituted (18) and (19) into (16) and (17), to determine the numbers ω_k, $\tilde{\omega}_k$, $\bar{\omega}_k$, and $\bar{\bar{\omega}}_k$ we obtain a system of algebraic equations*

$$\left[J_{\frac{n+a_0}{m}} - J_{\frac{a_0-n}{m}}\right]\left[\sum_{k=1}^{s}\omega_k\cos\frac{2k-1}{m}n\pi\right] + \left[J_{\frac{n+a_0}{m}} + J_{\frac{a_0-n}{m}}\right] \times \left[\sum_{k=1}^{s}\tilde{\omega}_k \sin\frac{2k-1}{m}n\pi\right] = \bar{A}_{-n} - \bar{A}_n, \tag{23}$$

where n = 1, 2, 3, . . . , m; m = 2s or m = 2s−1; s−1, 2, 3, . . . , N:

$$\left[J_{\frac{a_0-n}{m}} - J_{\frac{a_0+n}{m}}\right]\left[\sum_{k=1}^{s}\bar{\bar{\omega}}_k \sin\frac{2k-1}{m}n\pi\right] - \left[J_{\frac{n+a_0}{m}} + J_{\frac{a_0-n}{m}}\right]\left[\sum_{k=1}^{s}\bar{\omega}_k\cos\frac{2k-1}{m}n\pi\right] = \bar{B}_n + \bar{B}_{-n}, \tag{24}$$

where n = 0, 1, 2, 3, . . . , m−1; m = 2s or m = 2s−1; s = 1, 2, 3, . . . , N.

2. Examples of the Solutions for Certain Problems of Mathematical Physics

Example 1. Let us consider the special case of problems (7) and (8) for m = 2. Let $f_1(\theta) = b_0/2 + b_1 \cdot \cos\theta$, and the solution of Eq. (13) will then be

$$A_n = \bar{A}_n + \left(\omega_1 \cos\frac{n\pi}{2} + \tilde{\omega}_1 \sin\frac{n\pi}{2}\right) J_{\frac{n+a_0}{2}}\left(\frac{a_1}{2}\right). \tag{25}$$

From the recurrence relationships (21) we obtain

$$\bar{A}_0 = \bar{A}_1 = \bar{A}_2 = \cdots 0; \quad \bar{A}_{-1} = \frac{2b_1}{a_1}; \ \bar{A}_{-2} = \frac{2b_0}{a_1}. \tag{26}$$

*We can demonstrate that the determinants of systems (23) and (24) are not equal to zero.

The constants ω_1 and $\tilde{\omega}_1$ are determined directly from (23)

$$\omega_1 = \frac{b_0}{a_1} \frac{2}{J_{\frac{a_0}{2}-1} - J_{\frac{a_0}{2}+1}},$$
$$\tilde{\omega}_1 = \frac{b_1}{a_1} \frac{2}{J_{\frac{a_0-1}{2}} + J_{\frac{a_0+1}{2}}}. \tag{27}$$

Thus solution (12) can be written in the form

$$u = \omega_1 \left\{ \frac{1}{2} J_{\frac{a_0}{2}} \left(\frac{a_1}{2} \right) + \sum_{k=1}^{\infty} (-1)^k J_{k+\frac{a_0}{2}} \left(\frac{a_1}{2} \right) \rho^{2k} \cos 2k\theta \right\} + \tilde{\omega}_1 \left\{ \sum_{k=0}^{\infty} (-1)^k J_{k+\frac{a_0+1}{2}} \left(\frac{a_1}{2} \right) \rho^{2k+1} \cos (2k+1)\theta \right\}, \tag{28}$$

where $a_0 = hS/2\pi$; $a_1 = hS\lambda/\pi(1+\lambda^2)$; b_0 and b_1 are certain specified numbers; S is the arc length; h is the heat-transfer coefficient.

Example 2. Let us find the steady-state temperature distribution within the domain bounded by curve (5a), if uniform heat release is taking place within that region. The heat is radiated according to Newton's law at the boundary. The temperature of the external medium is equal to zero. The problem reduces to the integration of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{Q}{K} \text{ in } D \tag{29}$$

for the boundary condition

$$\left. \frac{\partial u}{\partial n} + hu \right|_{\Gamma} = 0, \tag{30}$$

where K is the coefficient of thermal conductivity and Q is the quantity of heat released per unit volume.

Let us isolate the particular solution

$$u = u_1 + u_2, \text{ где } \Delta u_2 = -\frac{Q}{K} \text{ in } D; \; u_2|_{\Gamma} = 0. \tag{31}$$

Then, for u_1 we have

$$\Delta u_1 = 0 \text{ in } D; \left. \frac{\partial u_1}{\partial n} + hu_1 \right|_{\Gamma} = -\left. \frac{\partial u_2}{\partial n} \right|_{\Gamma}. \tag{32}$$

We will use conformal mapping (6) and isolate the particular solution (31) of the problem, i.e.,

$$u_3 = -\frac{Q}{4K}(x^2 + y^2) = -\frac{Q}{4K} W\overline{W}. \tag{33}$$

Then, for u_2 we obtain

$$u_2 = -\frac{QS^2}{16\pi^2 K} \frac{\lambda^4}{(1+\lambda^2)^2} \left\{ \left[\rho^2 - 1 + \frac{4(\rho^{2m+2}-1)}{\lambda^2 (m+1)^2} + \frac{\rho^{4m+2}-1}{\lambda^4 (2m+1)^2} \right] \right.$$
$$\left. + \left[\frac{4(\rho^{m+2}-\rho^m)}{\lambda(m+1)} + \frac{4(\rho^{3m+2}-\rho^m)}{\lambda^3 (m+1)(2m+1)} \right] \cos m\theta + \frac{2(\rho^{2m+2}-\rho^{2m})}{\lambda^2 (2m+1)^2} \cos 2m\theta \right\}. \tag{34}$$

With the conformal mapping (6), the equation and boundary conditions (32) are brought to the form

$$\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u_1}{\partial \rho} \right) + \frac{\partial^2 u_1}{\partial \theta^2} = 0 \tag{35}$$

in a circle for the boundary conditions

$$\left. \frac{\partial u_1}{\partial \rho} + \frac{hS}{2\pi} \left(1 + \frac{2\lambda}{1+\lambda^2} \cos m\theta \right) u_1 \right|_{\rho=1} = \frac{QS^2}{8\pi^2 K} \left\{ \frac{\lambda^4 + \frac{4\lambda^2}{m+1} + \frac{1}{2m+1}}{(1+\lambda^2)^2} + \frac{\frac{4\lambda^3}{m+1} + \frac{4\lambda}{2m+1}}{(1+\lambda^2)^2} \cos m\theta \right.$$
$$\left. + \frac{2\lambda^2}{(2m+1)(1+\lambda^2)^2} \cos 2m\theta \right\}. \tag{36}$$

The solution for (35) and (36) will be sought in the form

$$u_1 = \frac{QS^2}{8\pi^2 K}\left[\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \rho^{mn} \cos mn\theta\right]. \tag{37}$$

The coefficients A_n are determined from the equation

$$(n + a_0) A_n + \frac{a_1}{2}[A_{n+1} + A_{n-1}] = b_n, \quad n = 0, 1, 2, \ldots, \tag{38}$$

for the conditions $A_n \underset{n\to\infty}{\to} 0$ and $A_{-1} = A_1$, where

$$a_0 = \frac{hS}{2\pi m}; \quad a_1 = \frac{hS}{\pi m}\,\frac{\lambda}{1+\lambda^2}, \quad m = 1, 2, 3, \ldots, N;$$

$$b_0 = \frac{4\lambda^4 + \dfrac{8\lambda^2}{m+1} + \dfrac{2}{2m+1}}{m(1+\lambda^2)^2}; \quad b_1 = \frac{\dfrac{4\lambda^3}{m+1} + \dfrac{4\lambda}{2m+1}}{m(1+\lambda^2)}; \tag{39}$$

$$b_2 = \frac{2\lambda^2}{m(2m+1)(1+\lambda^2)^2}; \quad b_3 = b_4 = \cdots = 0;$$

the solution of (38) can be derived from the recurrence relationships (21) and (22), from which we have

$$\bar{A}_1 = \frac{2}{a_1} b_2; \quad \bar{A}_0 = \frac{2}{a_1} b_1 - \left(\frac{2}{a_1}\right)^2 (a_0 + 1) b_2;$$

$$\bar{A}_{-1} = \frac{2b_0}{a_1} - \left(\frac{2}{a_1}\right)^2 a_0 b_1 + \left(\frac{2}{a_1}\right)^3 a_0 (a_0 + 1) b_2 - \frac{2b_2}{a_1}; \tag{40}$$

$$\bar{A}_2 = \bar{A}_3 = \bar{A}_4 = \cdots = 0.$$

Thus from (19), (23), and (40) we obtain

$$u_1 = \frac{QS^2}{8\pi^2 K}\left\{\frac{b_1}{a_1} - \frac{1}{2}\left(\frac{2}{a_1}\right)^2 (a_0 + 1) b_2 + \frac{2b_2}{a_1}\rho^m \cos m\theta \right.$$
$$\left. + \omega_1\left[\frac{1}{2} J_{a_0}(a_1) + \sum_{n=1}^{\infty} (-1)^n J_{n+a_0}(a_1) \rho^{mn} \cos mn\theta\right]\right\}, \tag{41}$$

where m = 1, 2, 3, . . . , N;

$$\omega_1 = \frac{\dfrac{2b_0}{a_1} - \left(\dfrac{2}{a_1}\right)^2 a_0 b_1 + \left(\dfrac{2}{a_1}\right)^3 a_0 (a_0 + 1) - \dfrac{4b_2}{a_1}}{2J'_{a_0}(a_1)}; \tag{42}$$

$J_{a_0}(a_1)$ is a Bessel function of the first kind. Solutions (41) and (42) are a generalization of the solution, derived earlier in [3], for the case m = 1.

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